Journal of Algebra 224, 467–478 (2000)
doi:10.1006/jabr.1999.8079, available online at http://www.idealibrary.com on IDEAL®

Finite Groups with Some Subgroups of Sylow Subgroups c-Supplemented

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Communicated by Walter Feit

Received January 8, 1999

A subgroup H is said to be c-supplemented in group G if there exists a subgroup K of G such that $HK = G$ and $H \cap K$ is contained in $Core_G(H)$. We determine the structure of finite groups with some subgroups of Sylow groups c-supplemented in G. © 2000 Academic Press

Key Words: solvable groups; Sylow subgroups; c-supplemented subgroups.

1. INTRODUCTION

It is of interest to use some information on Sylow subgroups of a group G to determine the structure of the group. In [S], Srinivassan proved that G is supersolvable if every maximal subgroup of Sylow subgroups is normal in G. The proof depends on a quite strong restriction that every Sylow p-subgroup is either normal or cyclic. Later Wall gave a complete classification for groups with such properties [Wa]. Srinivassan also tried to use a quasi-normal or subnormal condition to replace the normal condition. Let P_1 be a maximal subgroup of a Sylow p-subgroup. An elementary observation shows that P_1 is subnormal in G implies that $P_1 \leq F(G)$. Then it follows that either P_1 is a Sylow subgroup of $F(G)$ or $F(G)$ contains a Sylow p-subgroup of G. In both cases we have either the Sylow p-subgroup of G is normal in G or P_1 is normal in G. Quasi-normal follows subnormal in solvable groups [D-H, p. 234] and in this case in fact it follows normal. Hence the above normality assumptions are not significantly different from

[1] The author was supported in part by the NSF of China and NSFG.

0021-8693/00 $35.00

the normal assumption. The above conditions are definitely too strong to get the supersolvability or Sylow tower property of a group. In [W], Wang proved that one can replace the normal condition with the c-normal condition.

In this paper, we remove the normal supplement condition and replace the c-normal assumption with the c-supplement assumption for the maximal subgroups of Sylow groups of G. We get the supersolvability of G and some related results.

All the groups in this paper are finite.

Let π be a set of primes. We say that $G \in E_\pi$ if G has a Hall π-subgroup. We say that $G \in C_\pi$ if any two Hall π-subgroups of G are conjugate in G. We say that $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a Hall π-subgroup of G.

DEFINITION 1.1. *We say a subgroup H of a group G is c-supplemented (in G) if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G = Core_G(H)$. We say that K is a c-supplement of H in G.*

Recall that a subgroup H of G is said to be c-normal (in G) if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$ [W]. A subgroup H is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Hence c-supplementation is a generalization of c-normality and complementation, that is, to remove the normal supplementation assumption in c-normal and to remove the trivial intersection assumption in complementation.

c-supplementation cannot imply c-normality.

EXAMPLE 1. $A_5 = C_5 \cdot A_4$ and $C_5 \cap A_4 = 1$. Both C_5 and A_4 are complemented and so c-supplemented in A_5 but neither of them is c-normal since A_5 is simple and hence it is c-simple.

c-supplementation cannot imply complementation.

EXAMPLE 2. Let G be a finite cyclic p-group with order greater than p. Then $\Phi(G)$ is the only maximal subgroup of G which is clearly not complemented but it is normal and hence is c-supplemented.

Recently, in [B-G], the authors proved some nice results on the complementation of subgroups. In this paper, we generalize the main results of [B-G] with similar proofs. In fact, Theorem 3.3, Corollary 3.4, and Theorem 4.2 are generalizations of Theorem 2, Theorem 3, and Corollary 3 of [B-G].

2. ELEMENTARY PROPERTIES

The following lemma shows the basic properties of c-supplemented subgroups and it is very helpful in our later proofs.

LEMMA 2.1. *Let G be a group. Then*

(1) *If H is c-supplemented in G, $H \leq M \leq G$, then H is c-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is c-supplemented in G if and only if H/N is c-supplemented in G/N.*

(3) *Let π be a set of primes. Let N be a normal π'-subgroup and let H be a π-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N. If furthermore N normalizes H, then the converse also holds.*

(4) *Let $H \leq G$ and $L \leq \Phi(H)$. If L is c-supplemented in G, then $L \lhd G$ and $L \leq \Phi(G)$.*

Proof. (1) If $HK = G$ with $H \cap K \leq H_G$, then $M = M \cap G = H(M \cap K)$ and $H \cap (K \cap M) \leq H_G \cap M \leq H_M$. So H is c-supplemented in M.

(2) Suppose that H/N is c-supplemented in G/K. Then there exists a subgroup K/N of G/N such that $G/N = (H/N)(K/N)$ and $(H/N) \cap (K/N) \leq (H/N)_{G/N}$. It is easy to see that $G = HK$ and $H \cap K \leq H_G$.
Conversely, if H is c-supplemented in G, then there exists $K \leq G$ such that $G = HK$ and $H \cap K \leq H_G$. It is easy to check that KN/N is a c-supplement of H/N in G/N.

(3) If H is c-supplemented in G, then there exists $K \leq G$ such that $G = HK$ and $H \cap K \leq H_G$. Since $|G|_{\pi'} = |K|_{\pi'} = |KN|_{\pi'}$, we have that $|K \cap N|_{\pi'} = |N|_{\pi'} = |N|$ and hence $N \leq K$. It is clear that $(HN/N)(K/N) = G/N$ and $(HN/N) \cap (KN/N) = (H \cap K)N/N \leq (HN/N)_{G/N}$. Hence HN/N is c-supplemented in G/N.
Conversely, assume that HN/N is c-supplemented in G/N. Let K/N be a c-supplement of HN/N. Then $HK = HNK = G$ and $(H \cap K)N/N \leq L/N = ((HN)/N)_{G/N}$. By hypothesis, $NH = N \times H$. This means NH is both π-nilpotent and π-closed and hence $L = H_1 \times N$ with $H_1 \leq H$ and $H_1 \lhd G$. Now we have $H \cap K \leq H_1 \leq H_G$ and H is c-supplemented in G.

(4) In fact, if L is c-supplemented in G with supplement K, then $LK = G$ and $L \cap K \leq L_G$. Now $H = H \cap G = L(H \cap K) = H \cap K$ since $L \leq \Phi(H)$. Therefore $L \leq H \cap K \leq L_G$ and hence $L \trianglelefteq G$. If $L \nleq \Phi(G)$, then there exists a maximal subgroup M of G such that $LM = G$. Now $H = H \cap G = L(H \cap M) = H \cap M \leq M$. Therefore $G = LM \leq HM \leq M < G$, a contradiction.

THEOREM 2.2. *Let G be a finite group and let P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$. Suppose that there exists a maximal subgroup P_1 of P such that P_1 is a c-supplement in G. Then G is not a nonabelian simple group. Furthermore, if $(|G|, p-1) = 1$, then $G \in E_{p'}$.*

Proof. (1) G is not a nonabelian simple group.

Assume that G is a nonabelian simple group. By assumption there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_G = Core_G(P_1) = 1$. In particular, $[G : K] = p^r, r \geq 1$, and $p^r < |P| = p^{r+1}$. By [A-F, Theorem 5.8], either K is a Hall r'-subgroup of G or G is isomorphic to $PSU(4,2) \cong Psp(4,3)$ with $p^r = 3^2$ or G is isomorphic to A_n with $5 \leq n = p^r$, $r \geq 2$, and $K \cong A_{n-1}$. Clearly, K is not a Hall r'-subgroup of G since $[G : K] = p^r < |P| = p^{r+1}$. If $G \cong PSU(4,2)$, then $|G| = 25{,}920 = 2^6 \cdot 3^4 \cdot 5$ and $K = 2^6 \cdot 3^2 \cdot 5$ and $|P_1| = 3^2$ since $[G : K] = |P_1|/|P_1 \cap K|$ and by the definition of c-supplement, $P_1 \cap K \leq Core_G(P_1) = 1$, since we assume that G is nonabelian simple. But, in this case, $|P_1| = n = p^r \cdot (n!/2) = (1 \cdot 2 \cdot \cdots \cdot p^r)/2$. If $r > 1$, then $p^2 \mid |A_{n-1}|$ and $p^2 \mid [P : P_1]$, a contradiction since by assumption $[P : P_1] = p$. Therefore, G is not a nonabelian simple group.

(2) If $(|G|, p-1) = 1$, then $G \in E_{p'}$.

We prove this by induction on the order of G.

If $1 < (P_1)_G$, then $G/(P_1)_G$ satisfies the hypotheses and $G/(P_1)_G \in E_{p'}$ implies that $G \in E_{p'}$ by the Schur–Zassenhaus theorem [R, 9.1.10].

Now we assume that $(P_1)_G = 1$. Since P_1 is c-supplemented in G, there exists a subgroup K_1 of G such that $P_1 \cap K_1 \leq (P_1)_G \leq O_p(G) = 1$. Now $|K_1|_p = p$. Let K_{1p} denote a Sylow p-subgroup of K_1. Then $N_{K_1}(K_{1p})/C_{K_1}(K_{1p})$ is isomorphic to a subgroup of $Aut(K_{1p})$. Hence the order of $N_{K_1}(K_{1p})/C_{K_1}(K_{1p})$ must divide $(|G|, p-1) = 1$. Therefore $N_{K_1}(K_{1p}) = C_{K_1}(K_{1p})$. By Burnside's p-nilpotent theorem it follows that K_1 is p-nilpotent. It is clear that the normal p-complement $K_{1p'}$ is a Hall p'-subgroup of G and so $G \in E_{p'}$.

The theorem is proved.

If we only allow for one maximal subgroup of the Sylow 2-subgroup, we have the following theorem.

THEOREM 2.3. *Let G be a finite group and let P be a Sylow 2-subgroup of G. Suppose that there exists a maximal subgroup P_1 of P such that P_1 is c-supplemented in G. Then $G \in D_{2'}$ and G is not a nonabelian simple group and every composition factor is either a cyclic group of prime order or $B = PSL(2, r)$ for a Mersenne prime r and the Hall $2'$-subgroup of B is the normalizer of the Sylow q-subgroup of B.*

Proof. Gross' theorem shows that $E_{2'}$ implies $C_{2'}$ ([Gr, main theorem]. By [A-F, p. 547], note that, if π is a set of odd primes and G satisfies E_π

and $E_{\pi'}$, then $G \in D_{\pi}$. So if we can prove that $G \in E_{2'}$, then we have that $G \in D_{2'}$.

We prove that $G \in E_{2'}$, by induction on the order of G. If $1 < (P_1)_G$, then $G/(P_1)_G$ satisfies the hypotheses and every composition factor is either in $G/(P_1)_G$ or in $(P_1)_G$. The theorem is proved in this case.

Now we assume that $(P_1)_G = 1$. Since P_1 is c-supplemented in G, there exists a subgroup K of G such that $P_1 \cap K_1 \leq (P_1)_G = 1$. Now $|K|_p = 2$ and hence K is 2-nilpotent. Let K_2 denote the Sylow p-subgroup of K. It is clear that the normal 2-complement $K_{2'}$ is a Hall $2'$-subgroup of G and so $G \in E_{2'}$. Now we have $G = PK_{2'}$ and $K_{2'} < K \leq N_G(K_{2'})$. Assume that G is a nonabelian simple group. Then, by [A-F, Corollary 5.6(I)], G is isomorphic to $PSL(2, r)$ with r a Mersenne prime. Now [A-F, Theorem 5.8(II)] implies that $K = K_{2'} = N_G(G_r)$ since both of them have index of power 2, a contradiction. Since every composition factor of G also lies in $E_{2'}$, the conclusion follows from [A-F, Corollary 5.6 and Theorem 5.8].

THEOREM 2.4. *Let G be a finite group. Then G is solvable if and only if every Sylow subgroup of G is c-supplemented in G.*

Proof. If G is solvable, then, by [H, main theorem], every Sylow subgroup of G is complemented and hence is c-supplemented.

Conversely, assume that every Sylow subgroup P of G is c-supplemented in G. By [H, theorem], we only need to prove that P is complemented in G. Let K_1 be the c-supplement of P in G. Then $PK_1 = G$ and $P \cap K_1 \leq P_G$. Let $K = P_G K_1$. We have $PK = G$ and $P \cap K = P_G(P \cap K_1) = P_G$. Note that $|G|_p = (|P|\,|K|_p)/|P_G|$. So P_G is a normal Sylow p-subgroup of K. By the Schur–Zassenhaus theorem [R, Theorem 9.1.10], we have that $K = P_G K_{p'}$ with $K_{p'}$ the Hall p'-subgroup of K. Now $G = PK = PK_{p'}$ and $P \cap K_{p'} = 1$. Hence P is complemented in G.

The theorem is now proved.

3. MAIN RESULTS

In general, we cannot say too much if we only assume one Sylow subgroup is c-supplemented in G since we can easily find simple groups that satisfy the hypotheses. (In fact, we can find Sylow p-subgroups with complement in a simple group.) However, if we assume that every maximal subgroup of a Sylow subgroup is c-supplemented in G, then we can get much stronger results. We will show that we can get the same results for supersolvability of a group by simply replacing the normal assumption with the c-supplemented assumption (cf. Theorem 3.3). As we mentioned in the Introduction, both complementation and c-normality are the special cases of c-supplementation, and hence our results can be applied in very general cases.

THEOREM 3.1. *Let G be a finite group and let P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-supplemented in G and G satisfies $C_{p'}$. Then $G/O_p(G)$ is p-nilpotent and $G \in D_{p'}$.*

Proof. Assume that the theorem is false and choose for G a counterexample of smallest order. By Theorem 2.2 we have $G \in E_{p'}$. Furthermore we have:

(1) $O_p(G) = 1$.

If $O_p(G) = P$, then $G/O_p(G)$ is a p'-group and of course it is p-nilpotent, a contradiction. If $1 < O_p(G) < P$, then $G/O_p(G)$ satisfies the hypotheses and the minimal choice implies that $G/O_p(G) \cong (G/O_p(G))/O_p(G/O_p(G))$ is p-nilpotent, a contradiction.

(2) For every maximal subgroup P_1 of P, the c-supplement of P_1 is p-nilpotent.

Let P_1 be a maximal subgroup of P. By the hypotheses, P_1 is c-supplemented in G. So there exists a subgroup K_1 of G such that $P_1 \cap K_1 \leq (P_1)_G \leq O_p(G) = 1$. Now $|K_1|_p = p$. Let K_{1p} denote the Sylow p-subgroup of K_1. Then $N_{K_1}(K_{1p})/C_{K_1}(K_{1p})$ is isomorphic to a subgroup of $Aut(K_{1p})$. Hence the order of $N_{K_1}(K_{1p})/C_{K_1}(K_{1p})$ must divide $(|G|, p-1) = 1$. Therefore $N_{K_1}(K_{1p}) = C_{K_1}(K_{1p})$. Burnside's p-nilpotent theorem [Hu, Haupsatz IV.2.6] implies that K_1 is p-nilpotent.

(3) G is p-nilpotent.

Let P_1 be a maximal subgroup of P. By (1) and (2), there exists a subgroup K_1 of G such that $G = P_1K_1$ with $P_1 \cap K_1 = 1$ and K_1 is p-nilpotent. Let $K_1 = K_{1p}K_{1p'}$ and let $N = N_G(K_{1p'})$. We know that $N \neq G$; otherwise G is p-nilpotent, contrary to our choice. If $P \leq N$, then $N = G$, a contradiction. So we may assume that $P \cap N < P$. We can choose a maximal subgroup P_2 of P such that $P \cap N \leq P_2$. By the hypotheses, P_2 is c-supplemented. (2) implies that the supplement K_2 of P_2 is p-nilpotent. We denote $K_2 = K_{2p}K_{2p'}$. Now both $K_{1p'}$ and $K_{2p'}$ are Hall p'-subgroups of G. Since $G \in C_{p'}$, these two subgroups are conjugate in G. Say $(K_{1p'}) = (K_{2p'})^g$. Since $G = P_2K_2$ and $K_{2p'} \trianglelefteq K_2$ we may choose $g \in P_2$. We also have that K_2^g normalizes $K_{2p'}^g = K_{1p'}$ and hence $(K_2)^g \leq N$. Now $G = G^g = (P_2K_2)^g = P_2N$. Therefore $P = P \cap G = P_2(P \cap N) \leq P_2$. Since $P \cap N \leq P_2$, we have that $P \leq P_2$, a contradiction.

Combining (2) and (3), we conclude that there is no possible counterexample and the theorem is proved.

The following corollary is very useful in limiting our analysis to solvable groups. However, it involves the odd order theorem [F-T] and a very deep result of Gross [Gr].

COROLLARY 3.2. *Let G be a finite group and let P be a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. Suppose every maximal subgroup of P is c-supplemented in G. Then $G \in D_{p'}$ and $G/O_p(G)$ is a solvable p-nilpotent.*

Proof. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime divisor of $|G|$. If $p = 2$, then Gross' theorem shows that $E_{2'}$ implies $C_{2'}$ [Gr, main theorem]. If $p > 2$, then the odd order theorem implies that $G \in C_{p'}$. In both cases, the odd order theorem implies the solvability [F-T].

The following theorem generalizes the main theorem of [S] and some related results. We have proved that G is solvable if and only if every Sylow subgroup of G is c-supplemented in G. The following theorem shows that G will be supersolvable if we put the c-supplemented hypotheses on every maximal subgroup of a Sylow subgroup of G.

THEOREM 3.3. *Let G be a finite group and let N be a normal subgroup of G such that G/N is supersolvable. If every maximal subgroup of every Sylow subgroup of N is c-supplemented in G, then G is supersolvable.*

Proof. Assume that the theorem is false and choose G to be a counterexample of minimal order. Let r be the smallest prime divisor of $|N|$. By Lemma 2.1 we know that every maximal subgroup of the Sylow p-subgroup of N is c-supplemented in N. Corollary 3.2 implies that N is solvable and hence G is solvable. Let L be a minimal subgroup of G which is contained in N. Then L is an elementary abelian p-group for some prime p. Moreover, we have:

(1) G/L is supersolvable and L is the unique minimal normal subgroup of G which is contained in N. Furthermore, $L = F(N) = C_N(L)$.

First, we check that $(G/L, N/L)$ satisfies the hypotheses for (G, N). We know that $N/L \trianglelefteq G/L$ and $(G/L)/(N/L) \cong G/N$ is supersolvable. Let $\overline{Q} = QL/L$ be a Sylow q-subgroup of N/L. We may assume that Q is a Sylow q-subgroup of N. If $p = q$, we may assume that $L < P$. Then $P = Q \geq L$ and hence every maximal subgroup of $\overline{P}_1$ is of the form P_1/L with P_1 a maximal subgroup of P. By the hypotheses, P_1 is c-supplemented in G and hence P_1/L is c-supplemented in G/L by Lemma 2.1(2). Now we assume that $p \neq q$. Let $\overline{Q}_1$ be a maximal subgroup of a Sylow q-subgroup of $\overline{N}$ Without loss of generality, we may assume that $\overline{Q}_1 = Q_1L/L$ with Q_1 a maximal subgroup of a Sylow q-subgroup Q of N. Since Q_1 is c-supplemented in G, Lemma 2.1(3) implies that Q_1L/L is c-supplemented in G/L. So $(G/L, N/L)$ satisfies the hypotheses of the theorem. The minimal choice of G implies that G/L is supersolvable. Since the class of supersolvable groups is a saturated formation, we know that L is the unique minimal normal subgroup of G which is contained in N.

Since G/L is supersolvable, we have that $L \not\leq \Phi(G)$. So there exists a maximal subgroup M of G such that $G = LM$. Since $L \cap M$ is normalized by M and L, we have that $L \cap M \lhd G$. By the minimality of L, $L \cap M = 1$. Since $L \not\leq \Phi(F(N)) \leq \Phi(G)$, we have that $\Phi(F(N)) = 1$ and hence $F(N)$ is abelian. Now $F(N) = L(F(N) \cap M)$ and $F(N) \cap M$ is a normal subgroup of both M and $F(N)$ and hence it is normal in G. It follows that $F(N) \cap M = 1$ and so $F(N) = L$. N solvable implies that $F(N) \leq C_N(L) = C_N(F(N)) \leq F(N) = L$.

(2) L is a Sylow subgroup of N.

By (1), we have that G is solvable. Let q be the largest prime divisor of $|N|$ and let Q be a Sylow q-subgroup of N. Since N/L is a supersolvable subgroup of G/L. We have that $LQ/L \operatorname{Char} N/L \trianglelefteq G/L$ and so $LQ \trianglelefteq G$. If $p = q$, then $L \leq Q \lhd G$. Therefore $Q \leq F(N) = L$ and L is a Sylow q-subgroup of N.

Now we assume that $p < q$. Let P be the Sylow p-subgroup of N. Then $L \leq P$ and $PQ = PLQ$ is a subgroup of N. Note that every maximal subgroup of a Sylow subgroup of PQ is c-supplemented in G and hence is c-supplemented in PQ by Lemma 2.1(1). Therefore (PQ, PQ) satisfies the hypotheses for (G, N). If $PQ < G$, the minimal choice implies that PQ is supersolvable; in particular, $Q \trianglelefteq PQ$. Hence $LQ = L \times Q$ and $Q \leq C_N(L) \leq L$, a contradiction.

Now we assume that $G = PQ = N$ and $L < P$. Since N/L is supersolvable, $LQ \operatorname{char} N \lhd G = PQ$. By the Frattini argument, $G = LN_G(Q)$. Note that $L \cap N_G(Q)$ is normalized by $N_G(Q)$ and L. We have that $L \cap N_G(Q) = 1$ since L is the unique minimal normal subgroup of G which is contained in N and Q is not normal in G in this case. Therefore $G = L \rtimes N_G(Q)$. Let P_2 be a Sylow p-subgroup of $N_G(Q)$. Then LP_2 is a Sylow subgroup of G. Choose a maximal subgroup P_1 of LP_2 such that $P_2 \leq P_1$. Clearly, $N \not\leq P_1$ and hence $(P_1)_G = 1$. By our hypotheses, P_1 is c-supplemented in G. There exists a subgroup K of G such that $G = P_1K$ such that $P_1 \cap K \leq (P_1)_G = 1$. Now $|K|_p = |G : P_1|_p = p$, and since p is the smallest prime divisor of $|K|$ it follows that K has normal p-complement which is in fact a Sylow q-subgroup Q_1 of G in this case. By Sylow's theorem, there exists an element $g \in LP_2$ such that $Q_1^g = Q$. Since $P_1 \lhd LP_2$, we have that $G = P_1K = (P_1K)^g = P_1K^g$ and $P_1 \cap K^g = 1$. Since $K^g \cong K$ has normal p-complement and $Q = Q_1^g \leq K^g$, it follows that $K^g \leq N_G(Q)$. Since $LP_2 = LP_2 \cap G = P_1(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \not\leq P_2$. Otherwise $LP_2 \leq P_1P_2 \leq P_1$, a contradiction. Therefore P_2 is a proper subgroup of $P_3 = \langle P_2, LP_2 \cap K^g \rangle$ while P_3 is a subgroup of the p-Sylow subgroup LP_2.

Now both P_2 and K^g are contained in $N_G(Q)$ and we have that P_3 is a p-subgroup of G which contains a Sylow subgroup P_2 as a proper subgroup, a contradiction.

Hence L is a Sylow subgroup of N.

(3) $|L| = p$ and the final contradiction.

Let L_1 be a maximal subgroup of L. Then L_1 is c-supplemented in G. There exists a subgroup K of G with $L_1K = G$ and $L_1 \cap K \leq (L_1)_G = 1$. Therefore $L = L_1(L \cap K)$ and $L \cap K \trianglelefteq G$. We have that $L \cap K = N$ and hence $L \leq K$. Now since $|K|_p = |G : L_1|_p = |L : L_1| = p$, it follows that L is a cyclic group of prime order and hence G is supersolvable.

The final contradiction completes our proof.

COROLLARY 3.4. *Let G be a finite group. If every maximal subgroup of every Sylow subgroup of G' (respectively G) is c-supplemented in G, then G is supersolvable.*

We point out that the condition for c-supplementation of maximal subgroups of Sylow subgroups is sufficient for supersolvability but not necessary.

REMARK 3.5. *There exists a supersolvable group such that some of its maximal subgroups are not c-supplemented in G.*

EXAMPLE. Let K be a cyclic group of order 5 and let H be the full automorphism group of K of order 4. Let $G = K \rtimes H$. Then G is supersolvable but the maximal subgroup of H is not c-supplemented in G.

In fact, let $H = \langle \alpha \rangle$ with $|\alpha| = 4$. Since $\langle \alpha^2 \rangle$ is not normal in G (otherwise $\alpha^2 = 1$), we have that $\langle \alpha^2 \rangle_G = 1$. If $\langle \alpha^2 \rangle$ is c-supplemented in G, then it is complemented in P, which is contrary to $\langle \alpha^2 \rangle$ being the only maximal subgroup of H.

4. RESULTS FOR SECOND MAXIMAL SUBGROUPS

Now we will use the c-normality of second maximal subgroups of Sylow subgroups to characterize the structure of the group. A second maximal subgroup is a maximal subgroup of a maximal subgroup.

We first mention that A_5 is a simple group of order 60. The only possible second maximal subgroup of a Sylow 2-group is the identity group. It is normal in G. From the example, it seems that we cannot expect too much from the assumption of the c-supplementation of the second maximal subgroups of Sylow groups. However, the following results show that the reason is that A_5 contains A_4. If a group is A_4-free, the c-supplementation of the second maximal subgroups of Sylow subgroups can show in detail the structure of G.

LEMMA 4.1. *Let G be a finite group and let p be a prime divisor of $|G|$ such that $(|G|, p-1) = 1$. Assume that the order of G is not divisible by p^3 and G is A_4-free. Then G is p-nilpotent. In particular, if there exists odd prime p with $(|G|, p-1) = 1$ and the order of G is not divisible by p^3, then G is p-nilpotent.*

Proof. Assume the claim is false and let G be a counterexample of minimal order. Since every proper subgroup and every proper quotient group also satisfy the hypotheses of the theorem, the minimal choice of G follows that G is a non-p-nilpotent group but every proper subgroup and every proper quotient group of G is p-nilpotent. A well-known fact follows that G is a Schmidt group with normal p-subgroup. Therefore $G = P \rtimes Q$ with Q cyclic [R, 9.1.9]. Since both $\Phi(P)$ and $\Phi(G)$ are in $Z(G) = 1$, we have that P is an elementary abelian Sylow p-subgroup and Q a cyclic group of order q. $Q \cong G/P = N_G(P)/C_G(P)$ isomorphic to a subgroup of $Aut(P)$. Hence q divides $p(p+1)(p-1)$ (cf. [R, 3.2.7]). Since $p \neq q$ and $(p-1, q) = 1$, we have that $p = 2$ and $q = 3$ and $G \cong A_4$, a contradiction.

In the latter case, G is of odd order and hence it is A_4-free. The claim holds.

THEOREM 4.2. *Let G be a finite group and let p be the smallest prime divisor of $|G|$. Assume that G is A_4-free and every second maximal subgroup of the Sylow p-subgroup of G is c-normal in G. Then $G/O_p(G)$ is p-nilpotent.*

Proof. Assume the claim is false and let G be a counterexample of minimal order.

(1) $O_p(G) = 1$.

If $O_p(G) = P$, then $G/O_p(G)$ is a p'-group and of course it is p-nilpotent, a contradiction. If $1 < O_p(G) < P$, then $G/O_p(G)$ satisfies the hypotheses and the minimal choice implies that $G/O_p(G) \cong (G/O_p(G))/O_p(G/O_p(G))$ is p-nilpotent, a contradiction.

(2) $|G|$ is divisible by p^3.

If $p^3 \nmid |G|$, then G is p-nilpotent by Lemma 4.1, a contradiction.

(3) For every maximal subgroup P_1 of a Sylow subgroup P of G, the c-supplement of P_1 is p-nilpotent.

Let P be a Sylow p-subgroup of G and let P_1 be a second maximal subgroup of P. By the hypotheses, P_1 is c-supplemented in G. So there exists a subgroup K_1 of G such that $P_1 \cap K_1 \leq (P_1)_G \leq O_p(G) = 1$. Now $|K_1|_p = p^2$ and K_1 is A_4-free. Lemma 4.1 implies that K_1 is p-nilpotent.

(4) G is p-nilpotent.

Let $N = N_G(K_{1p'})$. By (3), $K_1 \leq N$, so we have that $G = P_1K_1 = P_1N$. $N \neq G$ since G is not p-nilpotent. By [Sc, Theorem 13.2.5], there exists a

Sylow p-subgroup P^* of N such that $P = P_1P^*$ is a Sylow p-subgroup of G. If $P^* = P$, then $N = G$, a contradiction. If P^* is a maximal subgroup of P, then $|G : N| = p$. Since p is the smallest prime divisor of $|G|$, we have that N is a normal subgroup of G. N is p-nilpotent implies that G has normal p-complement, a contradiction. Consequently, we may assume that $P^* \leq P_2$ where P_2 is a second maximal subgroup of P. By the hypotheses, P_2 is c-supplemented. By (3), there exists a p-nilpotent subgroup K_2 of G such that $G = P_2K_2$ and $P_2 \cap K_2 \leq (P_2)_G = 1$. Let $K_{2p'}$ be the normal Hall p'-subgroup of K_2. Then both $K_{1p'}$ and $K_{2p'}$ are Hall p'-subgroups of G. Since p is the smallest prime divisor of $|G|$, [Gr, main theorem] or the odd order theorem implies that $K_{1p'}$ and $K_{2p'}$ are conjugate in G. Let $K_{1p'} = K_{2p'}^g$. Note that $G = P_2K_2$ and $K_{2p'} \trianglelefteq K_2$. We may assume that $g \in P_2$. Therefore $G = (P_2K_2)^g = P_2K_2^g$. We have that $(K_2)^g$ normalizes $K_{2p'}^g = K_{1p'}$ and so $K_2^g \leq N$. Hence $G = P_2N$. Since $P = P \cap G = P_2(P \cap N)$ and $P \cap N = P^* \leq P_2$ by our choice, it follows that $P = P_2$, a contradiction.

The final contradiction completes our proof.

COROLLARY 4.3. *Let G be a finite group of odd order and let p be the smallest prime divisor of $|G|$. Assume that every second maximal subgroup of the Sylow p-subgroup of G is c-normal in G. Then $G/O_p(G)$ is p-nilpotent.*

Proof. Since G is of odd order, we have that G is A_4-free.

ACKNOWLEDGMENT

The author is grateful to the helpful suggestions of the referee. In fact, the first part of the Theorem 2.2 was modified by the referee.

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